Spherically symmetric solutions of gauge theories in eight dimensions

1984 J. Phys. A: Math. Gen. 17 2867

(http://iopscience.iop.org/0305-4470/17/14/030)

J. Phys. A: Math. Gen. 17 (1984) 2867–2872. Printed in Great Britain

Spherically symmetric solutions of gauge theories in eight dimensions

D B Fairlie† and J Nuyts‡

† University of Durham, Durham, DH1 3LE, UK
‡ University of Mons, 7000, Mons, Belgium

Received 26 April 1984

Abstract. Linear relations among the fields of a gauge theory in eight dimensions derived from a secular equation invariant under a SO(7) subgroup are particularly interesting. We present here the spherical solutions corresponding to this case when the gauge group is SO(7) or SO(8). In the former case the solution can be made regular everywhere.

1. Introduction

Increased attention has been focused recently on theories, be they gauge type (Corrigan *et al* 1983, Fairlie and Nuyts 1984) or supersymmetric (Cremmer *et al* 1978, Englert *et al* 1983, Englert 1983), in dimension D higher than four. In two recent papers (Corrigan *et al* 1983, Fairlie and Nuyts 1984), we have outlined the general approach to the generalisation of the usual self-duality in four dimensions for gauge theories in higher dimensions. Linear relations among the fields F derived from the secular equation ($\mu, \nu, \rho, \sigma = 1, \ldots, 8$)

$$\lambda F_{\mu\nu} = T_{\mu\nu\rho\sigma} F_{\rho\sigma}, \tag{1.1}$$

where T is a constant, arbitrary, completely antisymmetric tensor and λ a non-zero eigenvalue, imply the equations of motion

$$D_\mu F_{\mu\nu} = 0 \tag{1.2}$$

as a consequence of the Bianchi identities

$$D_\mu \wedge F_{\rho\sigma} = 0 \tag{1.3}$$

where $\wedge$ stands for complete antisymmetry in μ, ρ and σ. We have stressed the particular importance of the stability group of T. When D is eight and T is invariant under an $\widetilde{\mathrm{SO}}(7)$ subgroup of SO(8) ($\widetilde{\mathrm{SO}}(7)$ is the covering group of SO(7)), we have obtained particularly nice sets of equations. Either ($A, B = 1, \ldots, 7$)

$$\Lambda^A_{\mu\nu} F_{\mu\nu} = 0 \tag{1.4}$$

or

$$\Omega^{AB}_{\mu\nu} F_{\mu\nu} = 0 \tag{1.5}$$

where the seven 8×8 matrices Λ and the 21 (A, B antisymmetric) 8×8 matrices Ω are defined in the appendix.

In a first approach aimed at finding the solutions of these generalised linear equations we have studied here the SO(7) spherically symmetric solutions for an SO(7) or SO(8) gauge theory in an eight-dimensional space.

2. Spherically symmetric solutions

Let A be the potentials of an SO(8) gauge theory in an eight-dimensional space (space group SO(8) acting on an eight-dimensional x). Using the spherical symmetry under an $\widetilde{\mathrm{SO}}(7)$ subgroup of both groups, the general form of A (antisymmetric in α and β)

$$A_{\mu}^{\alpha\beta} = F\Omega_{\alpha\beta}^{CD}\Omega_{\rho\mu}^{CD}x^{\rho} + G\Lambda_{\alpha\beta}^{C}\Lambda_{\rho\mu}^{C}x^{\rho} \tag{2.1}$$

depends only on two x-squared dependent functions: F which describes the 21 part in $\alpha\beta$ and G which describes the 7 part (lower case Greek indices run from 1 to 8 and Latin capitals from 1 to 7).

When

$$G=0 \tag{2.2}$$

the gauge group is effectively reduced to SO(7) only, so that SO(7) and SO(8) gauge theories will be treated together. When

$$G=2F \tag{2.3}$$

equation (2.1) is fully SO(8) spherically symmetric. Indeed, in this case

$$A_{\mu}^{\alpha\beta} = 2F(\delta_{\mu}^{\beta}x^{\alpha} - \delta_{\mu}^{\alpha}x^{\beta}). \tag{2.4}$$

The fields, defined by

$$F_{\mu\nu}^{\alpha\beta} = \partial_{\mu}A_{\nu}^{\alpha\beta} - \partial_{\nu}A_{\mu}^{\alpha\beta} + A_{\mu}^{\alpha\gamma}A_{\nu}^{\gamma\beta} - A_{\nu}^{\alpha\gamma}A_{\mu}^{\gamma\beta}, \tag{2.5}$$

can then be written in terms of five independent tensors, which may involve the 35 piece of the symmetrised product

$$X_{\rho\sigma} = x_{\rho}x_{\sigma} - \tfrac{1}{8}\delta_{\rho\sigma}x^{2}. \tag{2.6}$$

After some algebra, using the identities (A17)–(A24) of the appendix, one obtains

$$\begin{aligned} F_{\mu\nu}^{\alpha\beta} = {} & \Omega_{\alpha\beta}^{AB}\Omega_{\mu\nu}^{AB}(2F + \tfrac{1}{2}F'x^{2} - 5F^{2}x^{2} - \tfrac{1}{4}G^{2}x^{2}) \\ & + \Lambda_{\alpha\beta}^{A}\Lambda_{\mu\nu}^{A}(2G + \tfrac{1}{2}G'x^{2} - 6FGx^{2}) \\ & + X_{\rho\sigma}(\Omega_{\alpha\beta}^{AB}\Omega_{\mu\nu}^{CD}\Phi_{\rho\sigma}^{AB,CD})(-\tfrac{1}{4}F' - \tfrac{1}{2}F^{2} - \tfrac{1}{2}G^{2}) \\ & + X_{\rho\sigma}(\Omega_{\alpha\beta}^{AB}\Lambda_{\mu\nu}^{C}\Sigma_{\rho\sigma}^{C,AB})(-\tfrac{1}{2}F' - 3F^{2} + G^{2}) \\ & + X_{\rho\sigma}(\Lambda_{\alpha\beta}^{C}\Omega_{\mu\nu}^{AB}\Sigma_{\rho\sigma}^{C,AB})(-\tfrac{1}{4}G' - 2FG) \end{aligned} \tag{2.7}$$

where the prime denotes differentiation with respect to x squared.

There will be no 7 part in F (i.e. F satisfies (1.4)) if the two following equations are satisfied:

$$\left.\begin{aligned} & 2G + \tfrac{1}{2}G'x^{2} - 6FGx^{2} = 0, \\ & -\tfrac{1}{2}F' - 3F^{2} + G^{2} = 0, \end{aligned}\right\} \quad \text{(no 7)} \tag{2.8}$$

while there will be no 21 part in F (i.e. F satisfies (1.5)) if the three following equations are satisfied:

$$
\begin{aligned}
&2F+\tfrac{1}{2}F'x^2-5F^2x^2-\tfrac{1}{4}G^2x^2=0,\\
&-\tfrac{1}{4}F'-\tfrac{1}{2}F^2-\tfrac{1}{2}G^2=0, \qquad \text{(no 21)}\\
&-\tfrac{1}{4}G'-2FG=0.
\end{aligned}
\tag{2.9}
$$

If we look for solutions which behave asymptotically as an inverse square law

$$
\begin{aligned}
F&=f/x^2,\\
G&=g/x^2,
\end{aligned}
\tag{2.10}
$$

we obtain in the first case (no 7 part)

$$
\begin{aligned}
&g(1-4f)=0,\\
&f-6f^2+2g^2=0,
\end{aligned}
\qquad \text{(no 7)}
\tag{2.11}
$$

i.e. the two possibilities

$$
\text{(IA):} \qquad g=0, \qquad f=\tfrac{1}{6}, \tag{2.12}
$$

corresponding to an SO(7) gauge group or

$$
\text{(IB):} \qquad f=\tfrac{1}{4}, \qquad g=\pm\tfrac{1}{4}. \tag{2.13}
$$

However, in the second case (no 21 part) there is no such solution, as can easily be checked. This result could have been foreseen in the following way. Written with the complex variables

$$
\begin{aligned}
y&=x_1+\mathrm{i}x_2, \qquad & z&=x_3+\mathrm{i}x_4,\\
w&=x_5+\mathrm{i}x_6, \qquad & t&=x_7+\mathrm{i}x_8,
\end{aligned}
\tag{2.14}
$$

the 21 equations (A11) have the following three equations as a subset:

$$
F_{y\bar{z}}=F_{y\bar{w}}=F_{y\bar{t}}=0 \tag{2.15}
$$

together with their complex conjugates. From this one finds easily that A has to be a pure gauge,

$$
A_a=K^{-1}\partial_a K, \tag{2.16}
$$

i.e. that F must be zero. (In the proof one has to take into account that, since $\widetilde{\mathrm{SO}}(7)$ is democratically embedded in SO(8), by its very form (2.1) the A's for a given μ depend on all eight x.)

3. Conclusion

In this paper we have shown that there are $\widetilde{\mathrm{SO}}(7)$ spherically symmetric solutions of the secular equation (1.1), for gauge theories with gauge group SO(7) (see (2.11)) or SO(8) (see (2.12)) in an eight-dimensional space. It is, however, amusing to note that these solutions exist only for the 21-dimensional case (no 7), i.e. when F belongs to the space of the adjoint representation of the gauge group. One may wonder if this is a result valid in general. Remember that in the four-dimensional case both self-duality and anti-self-duality for the fields correspond to the restriction to one of the two SU(2) subgroups in SO(4), according to

$$\mathrm{SO}(4) = (\mathrm{SU}(2) \times \mathrm{SU}(2))/Z_2. \tag{3.1}$$

The solutions which we have found and which are singular at the origin can easily be generalised in two ways. First by shifting the origin at any place a,

$$x_\mu \to x_\mu - a_\mu. \tag{3.2}$$

The singularity can be removed altogether by the replacement

$$x^2 \to x^2 + k \tag{3.3}$$

with an arbitrary constant k in the case $g = 0$ (2.12).

Appendix

We have shown in Corrigan *et al* (1983) and Fairlie and Nuyts (1984) that the seven following linear relations in SO(8),

$$F_{81} + F_{72} + F_{45} + F_{36} = 0, \tag{A1}$$

$$F_{82} + F_{17} + F_{35} + F_{64} = 0, \tag{A2}$$

$$F_{83} + F_{74} + F_{52} + F_{61} = 0, \tag{A3}$$

$$F_{84} + F_{37} + F_{51} + F_{26} = 0, \tag{A4}$$

$$F_{85} + F_{76} + F_{14} + F_{23} = 0, \tag{A5}$$

$$F_{86} + F_{57} + F_{13} + F_{42} = 0, \tag{A6}$$

$$F_{87} + F_{65} + F_{43} + F_{21} = 0, \tag{A7}$$

imply the equations of motion through a secular equation (1.1) where T is invariant under $\widetilde{\mathrm{SO}}(7)$, with eigenvalue 1. Equations (A1)–(A7) can be written ($\mu\nu = 1, \ldots, 8$) ($B = 1, \ldots, 7$)

$$\Lambda^B_{\mu\nu} F_{\mu\nu} = 0 \tag{A8}$$

by defining the Bth antisymmetrix matrix Λ as having zeros everywhere except four times $+1$ when needed by (AB) and four times -1 by antisymmetry. In what follows capital Latin indices will run from 1 to 7 and lower case Greek indices from 1 to 8.

Through the seven equations (A8) the 28-dimensional F space is restricted to a 21-dimensional subspace. The orthogonal complement corresponds to the second eigenvalue -3. It is seven dimensional (21 linear relations),

$$\Omega^{AB}_{\mu\nu} F_{\mu\nu} = 0, \tag{A9}$$

where the Ω's are antisymmetric both in $\mu\nu$ and in AB,

$$\Omega^{AB}_{\mu\nu} = \tfrac{1}{2}(\Lambda^A_{\mu\gamma}\Lambda^B_{\gamma\nu} - \Lambda^B_{\mu\gamma}\Lambda^A_{\gamma\nu}). \tag{A10}$$

The equations (A9) are equivalent to the equations obtained from (A1)–(A7) by equating the F's appearing in them line by line, namely

$$F_{81} = F_{72} = F_{45} = F_{36} \tag{A11}$$

and six other relations of that form.

The Λ and Ω matrices can be thought of as being the Clebsch–Gordan coefficients for the couplings

$$8\times 8\to 7:\Lambda \tag{A12}$$

and

$$8\times 8\to 21:\Omega. \tag{A13}$$

Let us now recall the useful products of representations in $\widetilde{\mathrm{SO}}(7)$

$$\begin{aligned} &7\times 7=1_{\mathrm{s}}+21_{\mathrm{a}}+27_{\mathrm{s}}, \qquad 8\times 8=1_{\mathrm{s}}+7_{\mathrm{a}}+21_{\mathrm{a}}+35_{\mathrm{s}},\\ &7\times 21=7+35+105, \qquad 7\times 8=8+48,\\ &21\times 21=1_{\mathrm{s}}+21_{\mathrm{a}}+27_{\mathrm{s}}+35_{\mathrm{s}}+168_{\mathrm{s}}+189_{\mathrm{a}}, \end{aligned} \tag{A14}$$

and related Clebsch–Gordan coefficients

$$\begin{aligned} &7\times 21\to 35,\\ &\Sigma^{D,AB}_{\rho\sigma}=\tfrac{1}{2}((\Lambda^{D}_{\sigma\gamma}\Omega^{AB}_{\gamma\rho}+(\sigma\rho))-\mathrm{Tr}(\rho\sigma)) \end{aligned} \tag{A15}$$

where $+(\sigma\rho)$ symmetrises in ρ and σ and $-\mathrm{Tr}(\sigma\rho)$ extracts the trace part so as to make Σ symmetric and traceless in ρ and σ.

$$\begin{aligned} &21\times 21\to 35_{\mathrm{s}},\\ &\Phi^{AB,CD}_{\rho\sigma}=\tfrac{1}{2}((\Omega^{AB}_{\sigma\gamma}\Omega^{CD}_{\gamma\rho}+(\rho\sigma))-\mathrm{Tr}(\rho\sigma)). \end{aligned} \tag{A16}$$

The following identities have been used in the text:

$$(\Lambda^{A}_{\alpha\gamma}\Lambda^{A}_{\mu\delta}\Lambda^{B}_{\beta\gamma}\Lambda^{B}_{\nu\delta}-(\mu\nu))=2\Omega^{AB}_{\alpha\beta}\Omega^{AB}_{\mu\nu}, \tag{A17}$$

$$(\Omega^{AB}_{\alpha\gamma}\Omega^{AB}_{\mu\delta}\Omega^{CD}_{\beta\gamma}\Omega^{CD}_{\nu\delta}-(\mu\nu))=40\Omega^{AB}_{\alpha\beta}\Omega^{AB}_{\mu\nu}, \tag{A18}$$

$$((\Omega^{AB}_{\alpha\gamma}\Omega^{AB}_{\mu\delta}\Lambda^{C}_{\beta\gamma}\Lambda^{C}_{\nu\delta}-(\alpha\beta))-(\mu\nu))=48\Lambda^{A}_{\alpha\beta}\Lambda^{A}_{\mu\nu}, \tag{A19}$$

$$(((\Omega^{AB}_{\rho\mu}\delta_{\nu\sigma}-(\mu\nu))+(\rho\sigma))-\mathrm{Tr}(\rho\sigma))=\tfrac{1}{4}\Omega^{CD}_{\mu\nu}\Phi^{AB,CD}_{\rho\sigma}+\tfrac{1}{2}\Lambda^{D}_{\mu\nu}\Sigma^{D,AB}_{\rho\sigma}, \tag{A20}$$

$$(((\Lambda^{A}_{\rho\mu}\delta_{\nu\sigma}-(\mu\nu))+(\rho\sigma))-\mathrm{Tr}(\rho\sigma))=\tfrac{1}{4}\Omega^{CD}_{\mu\nu}\Sigma^{A,CD}_{\rho\sigma}, \tag{A21}$$

$$\begin{aligned} &(((\Lambda^{A}_{\alpha\gamma}\Lambda^{A}_{\mu\rho}\Lambda^{C}_{\beta\gamma}\Lambda^{C}_{\nu\sigma}-(\mu\nu))+(\rho\sigma))-\mathrm{Tr}(\rho\sigma))\\ &\qquad=\Omega^{AB}_{\alpha\beta}\Omega^{CD}_{\mu\nu}\Phi^{AB,CD}_{\rho\sigma}-2\Omega^{AB}_{\alpha\beta}\Lambda^{C}_{\mu\nu}\Sigma^{C,AB}_{\rho\sigma}, \end{aligned} \tag{A22}$$

$$\begin{aligned} &(((\Omega^{AB}_{\alpha\gamma}\Omega^{AB}_{\mu\rho}\Omega^{CD}_{\beta\gamma}\Omega^{CD}_{\nu\sigma}-(\mu\nu))+(\rho\sigma))-\mathrm{Tr}(\rho\sigma))\\ &\qquad=\Omega^{AB}_{\alpha\beta}\Omega^{CD}_{\mu\nu}\Phi^{AB,CD}_{\rho\sigma}+6\Omega^{AB}_{\alpha\beta}\Lambda^{C}_{\mu\nu}\Sigma^{C,AB}_{\rho\sigma}, \end{aligned} \tag{A23}$$

$$((((\Omega^{AB}_{\alpha\gamma}\Omega^{AB}_{\mu\rho}\Lambda^{C}_{\beta\gamma}\Lambda^{C}_{\nu\sigma}-(\alpha\beta))-(\mu\nu))+(\rho\sigma))-\mathrm{Tr}(\rho\sigma))=4\Lambda^{A}_{\alpha\beta}\Omega^{BC}_{\mu\nu}\Sigma^{A,BC}_{\rho\sigma}, \tag{A24}$$

as well as

$$\Lambda^{C}_{\alpha\beta}\Lambda^{D}_{\alpha\beta}=8\delta^{CD}, \tag{A25}$$

$$\Omega^{AB}_{\alpha\beta}\Omega^{CD}_{\alpha\beta}=8(\delta^{AC}\delta^{BD}-\delta^{AD}\delta^{BC}), \tag{A26}$$

$$\Lambda^{C}_{\alpha\beta}\Omega^{AB}_{\alpha\beta}=0. \tag{A27}$$

References

Corrigan E, Devchand C, Fairlie D B and Nuyts J 1983 *Nucl. Phys.* B **214** 452
Cremmer E, Julia B and Scherk J 1978 *Phys. Lett.* **76B** 409
Englert F 1983 *Phys. Lett.* **119B** 339
Englert F, Rooman M and Spindel Ph 1983 *Symmetries in eleven dimensional supergravity compactified on a parallelized seven sphere*
Fairlie D B and Nuyts J 1984 *J. Math. Phys.* **25** 2025